

SMI-15-97

# Phase of a chiral determinant and global $SU(2)$ anomaly.

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February 1, 2008

## Abstract

A representation for the phase of a chiral determinant in terms of a path integral of a local action is constructed. This representation is used to modify the action of chiral  $SU(2)$  fermions removing the global anomaly.

## 1 Introduction

It is known that  $SU(2)$  gauge models with an odd number of Weyl fermion doublets are affected by a global quantum anomaly leading to inconsistency of the theory [1]. This anomaly is related to an ambiguity in the definition of the phase of a chiral determinant. It was shown in the paper [1] that one cannot define the phase of the determinant of a single chiral  $SU(2)$  fermion in a gauge invariant way.

In the present paper I will get a representation for the phase of a chiral determinant as a path integral of exponent of a local action. Having this explicit representation one can modify the action of chiral  $SU(2)$  fermions in such a way that the global anomaly disappears. My construction in some sence is reminiscent to the Wess-Zumino construction for local quantum anomalies [2].

It also gives a representation for Atiyah-Patodi-Singer  $\eta$ -invariant [3] as a path integral of a local action, as it was shown before [4] that the phase of a chiral determinant is expressed in terms of  $\eta$ .

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## 2 Lagrangian representation for the phase of a chiral determinant and global $SU(2)$ anomaly

Let us start with the massless Euclidean Dirac operator

$$\hat{D} = \gamma_\mu (\partial_\mu + iA_\mu(x)) \quad (1)$$

which can be written in the form

$$\hat{D} = \begin{pmatrix} 0 & C \\ -C^+ & 0 \end{pmatrix} \quad (2)$$

where

$$C = e_\mu (\partial_\mu + iA_\mu(x)) \quad (3)$$

with  $e_i = -i\sigma_i$ ,  $\sigma_i$  ( $i = 1, 2, 3$ ) being the Pauli matrices,  $e_0 = -I$ . The field  $A_\mu(x)$  for each  $x$  belongs to the Lie algebra of a gauge group. The matrices  $C, C^+$  represent fermions of opposite chiralities.

It follows from the eq.(2) that

$$\det D = \det C \det C^+ \quad (4)$$

Hence the determinant of the Dirac operator is equal to the square of the modulus of the determinant of the Weyl operator. In case of the  $SU(2)$  group due to pseudoreality of representation

$$\det C = \det C^+ \quad (5)$$

Using this fact Witten proposed to define the regularized determinant of a single Weyl fermion as a square root of regularized Dirac determinant. Dirac determinant can be regularized in a gauge invariant way by means of a standard Pauli-Villars regularization. However there is a sign ambiguity

$$\det C_R = \pm (\det D_R)^{1/2} \quad (6)$$

For a given gauge field  $A_\mu(x)$  one can choose a sign arbitrary. Defined in such a way the Weyl determinant is obviously invariant with respect to infinitesimal gauge transformations and therefore in the framework of perturbation theory one has a consistent gauge invariant definition of the determinant of the chiral  $SU(2)$  operator.

However if one allows topologically nontrivial gauge transformations, then one can choose a transformation  $U$  which changes the sign of the square root of the determinant

$$[\det D(A_\mu)]^{1/2} = -[\det D(A_\mu^U)]^{1/2} \quad (7)$$

That means the phase of a chiral determinant is not invariant with respect to "large" gauge transformations and leads to inconsistency of the theory.

The definition of the chiral  $SU(2)$  determinant as a square root of the regularized Dirac determinant is perfectly consistent in the framework of perturbation theory, however it does not allow a representation of the Weyl determinant as a path integral

of a local action as taking a square root of a determinant is a nonlocal operation. To get such a representation for regularized chiral determinant we use the idea proposed in our paper [5].

We introduce the infinite set of Pauli-Villars fields with masses  $Mr$ ,  $r = 1, 2, \dots$ . Now the regularized Weyl determinant may be written as follows

$$\det C_R = \int \exp\{\int L_R dx\} d\bar{\psi}_+ d\psi_+ d\bar{\psi}_r d\psi_r \quad (8)$$

$$L_R = \bar{\psi}_+ C \psi_+ + \sum_{r=1}^{\infty} \bar{\psi}_r (\hat{D} + Mr) \psi_r$$

Here  $\psi_r$  are Pauli-Villars fields having Grassmannian parity  $(-1)^{r+1}$ .

Integrating over  $\psi$  we get

$$\det C_R = \det C \prod_{r=1}^{\infty} \det (\hat{D} + Mr)^{(-1)^r} \quad (9)$$

Using the representation like eq.(2) for the Dirac operator one can rewrite it as follows

$$\begin{aligned} \det C_R &= \det C \prod_{r=1}^{\infty} \det (|C|^2 + M^2 r^2)^{(-1)^r} = \\ &= \prod_{r=0}^{\infty} \prod_i C_i^{-1} (|C_i|^2 + M^2 r^2)^{(-1)^r} \end{aligned} \quad (10)$$

Here  $C_i$  are diagonal elements of the matrix  $C_{ij} = \langle u_i C v_j \rangle$ , where  $u_i$  and  $v_j$  form orthonormal bases in the spaces of left and right handed fermions respectively.

The product over  $r$  can be calculated explicitly using the representation

$$\prod_{r=0}^{\infty} (|C_i|^2 + M^2 r^2)^{(-1)^r} = \exp\{\sum_{r=0}^{\infty} \ln(|C_i|^2 + M^2 r^2) (-1)^r\} \quad (11)$$

Differentiating the exponent with respect to  $|C^i|^2$  one has

$$\begin{aligned} \frac{\partial}{\partial |C_i|^2} \sum_{r=0}^{\infty} (-1)^r \ln(|C_i|^2 + M^2 r^2) &= 1/2 \left[ \sum_{r=-\infty}^{\infty} (-1)^r (|C_i|^2 + M^2 r^2)^{-1} + |C_i|^{-2} \right] = \\ &= 1/2 [\pi (M |C_i| \sinh(\pi |C_i| M^{-1}))^{-1} + |C_i|^{-2}] \end{aligned} \quad (12)$$

Integrating over  $|C_i|^2$  one gets

$$\ln(\det C_R) = \ln(\tan(\frac{\pi |C_i|}{2M})) + \ln(|C_i|) + \ln(B) \quad (13)$$

where  $B$  is a field independent constant which in the following is assumed to be included into normalization factor. Therefore up to normalization factor we have the following representation for the regularized determinant

$$\det C_R = \prod_i \frac{|C_i|}{C_i} \tan(\frac{\pi |C_i|}{2M}) \quad (14)$$

In the framework of perturbation theory for a given  $A_\mu$  one can fix the signs of  $C_i$  at will, in particular take all  $C_i$  positive. Then it follows from the eq.(14) that

$$\det C_R = \prod_i \tan\left(\frac{\pi|C_i|}{2M}\right) \quad (15)$$

One sees that all  $|C_i| >> M$  are cutted and therefore eqs.(8, 15)indeed provide the gauge invariant regularization of the chiral determinant. It can be demonstrated explicitely in terms of Feynmann diagrams. In particular for polarization operator one has

$$\begin{aligned} \Pi_{ij,R}^{\mu\nu} = & \int \frac{d^4p}{(2\pi)^4} \text{tr} \left\{ \frac{(1+\gamma_5)}{2} \frac{\gamma_\mu \tau_i(\hat{p} - \hat{k}) \gamma_\nu \tau_j \hat{p}}{(p-k)^2 p^2} \right\} + \\ & + \sum_{r=1}^{\infty} \int \frac{d^4p}{(2\pi)^4} \text{tr} \left\{ \frac{\gamma_\mu \tau_i(\hat{p} - \hat{k} + Mr) \gamma_\nu \tau_j(\hat{p} + Mr) (-1)^r}{[(p-k)^2 + M^2 r^2][p^2 + M^2 r^2]} \right\} \end{aligned} \quad (16)$$

Using the standard technique one can rewrite it as follows

$$\begin{aligned} \Pi_{ij,R}^{\mu\nu} = & \delta_{ij} \int \frac{d^4p}{(2\pi)^4} \int_0^1 d\alpha [1/2 \frac{\text{tr} [\gamma_\mu(\hat{p} + \alpha \hat{k} - \hat{k}) \gamma_\nu(\hat{p} + \alpha \hat{k})]}{[p^2 + k^2(\alpha - \alpha^2)]^2}] + \\ & + \sum_{r=1}^{\infty} \frac{\text{tr} [\gamma_\mu(\hat{p} + \alpha \hat{k} - \hat{k} + Mr) \gamma_\nu(\hat{p} + \alpha \hat{k} + Mr)] (-1)^r}{[p^2 + k^2(\alpha - \alpha^2) + M^2 r^2]^2} \end{aligned} \quad (17)$$

Separating the terms which do not contain  $M$  in the nominator and performing the summation we get

$$\begin{aligned} \Pi_1 \sim & \int \frac{d^4p}{(2\pi)^4} 1/2 \int_0^1 d\alpha \text{tr} [\gamma_\mu(\hat{p} + \alpha \hat{k} - \hat{k}) \gamma_\nu(\hat{p} + \alpha \hat{k})] \times \\ & \left( -\frac{\partial}{\partial |p|^2} \pi(M \sqrt{p^2 + k^2(\alpha - \alpha^2)} \sinh(\pi \sqrt{p^2 + k^2(\alpha - \alpha^2)} M^{-1}))^{-1} \right) \end{aligned} \quad (18)$$

One sees that the integrand decreases exponentially for large  $p$  providing fast convergence of the integral. The terms which contain  $M$  in the nominator are analyzed in the same way.

However beyond perturbation theory the expression (14) is not well defined. Although large eigenvalues are cutted, the phase factor is not regularized. It is the source of the global anomaly in our approach. As was mentioned in the beginning, topologically nontrivial gauge transformations may change the sign of  $C_i$  and one cannot fix it in arbitrary way. (A different possibility to get the global anomaly starting from a local action was discussed in ref. [6], where embedding of the  $SU(2)$  model into anomalous  $SU(3)$  theory was considered).

It is known that in the case of local anomalies one can restore gauge invariance by adding to the action a local term depending on new fields [2]. Below I show that a similar construction is possible for the global anomaly as well.

Let us modify the Lagrangian (7) by adding to it a gauge invariant term describing the interaction of new fields  $\chi_r$

$$L_R \rightarrow L'_R = L_R + \Delta L$$

$$\Delta L = \bar{\chi}_+ C \chi_+ + \sum_{r=1}^{\infty} \bar{\chi}_r (\hat{D} + m^2 M^{-1} r) \chi_r \quad (19)$$

where  $\chi_r$  are again the fields with alternating Grassmannian parity  $(-1)^{r+1}$ . Here  $m$  is some fixed parameter with the dimension of mass. When  $M \rightarrow \infty$  the masses of the  $\chi$  fields vanish.

The integral over  $\chi$  can be calculated as above giving the result

$$\Delta = \prod_i \frac{|C_i|}{C_i} \tan\left(\frac{\pi |C_i|}{2m^2 M^{-1}}\right) \quad (20)$$

Assuming that  $C_i \neq 0$  we see that when  $M \rightarrow \infty$

$$\Delta \rightarrow \prod_i \frac{|C_i|}{C_i} \quad (21)$$

It shows that the integral

$$\lim_{M \rightarrow \infty} \int \exp\{\int \Delta L dx\} d\bar{\chi}_+ d\chi_+ d\bar{\chi}_r d\chi_r \quad (22)$$

gives the representation for the phase of a chiral determinant as a path integral of the local action. Note that this representation is valid for any gauge group, not necessary  $SU(2)$ .

One sees that the  $\Delta$  exactly compensates the indefinite phase factor in the eq.(14) and the integral

$$\det(C_R)' = \int \exp\{\int L'_R dx\} d\bar{\psi}_+ d\psi_+ d\bar{\chi}_+ d\chi_+ d\bar{\psi}_r d\psi_r d\bar{\chi}_r d\chi_r \quad (23)$$

provides a well defined expression which is gauge invariant not only with respect to infinitesimal gauge transformations but with respect to topologically nontrivial transformations as well.

In the case of the  $SU(2)$  gauge group the new fields  $\chi$  do not influence the results of perturbative calculations as in this case one can fix arbitrary the signs of  $C_i$  and  $\lim_{M \rightarrow \infty} \Delta = 1$ . It is also seen from the explicit calculation of the polarization operator. When the mass of the  $\chi$  fields tends to zero the integral vanishes.

Analogous construction is valid for other models with global anomalies, provided the fermions belong to a pseudoreal representation.

### 3 Discussion

We constructed above a representation for the phase of a chiral determinant in terms of a path integral of the exponent of the local action. Using this expression we were able to modify the action of  $SU(2)$  chiral fermions in such a way that the global anomaly disappears. This construction has some similarity to the Wess-Zumino construction for local anomalies. There are however important differences. The Wess-Zumino term restores quantum gauge invariance, but the classical action

including this term is not gauge invariant, whereas our modified action (19) is gauge invariant. Another difference is that contrary to the Wess-Zumino case, variation of our additional term under topologically nontrivial gauge transformation is discrete. The geometrical meaning of these terms is also different. As was mentioned above our construction gives a representation for Atiyah-Patodi-Singer  $\eta$ -invariant.

It would be interesting to analyze a possible physical meaning of the modified action. Although the new fields are not seen in perturbation theory, they certainly influence nonperturbative configurations. Even if one starts with the configuration which does not include the  $\chi$  fields, they will be produced in pairs in a final state. One can speculate that the true vacuum of the modified model includes infinite number of pairs of massless fermions which may change drastically a physical content of the theory.

#### **Acknowledgements.**

This work was completed while the author was participating in the session of Bernalasque summer institute for theoretical physics. I am grateful to the organizers and in particular to A.Gonzales-Arroyo for invitation and kind hospitality extended to me in Spain. I acknowledge useful discussions with L.Alvarez-Gaume and M.Asorey. This research is supported in part by Russian Basic Research Foundation under grant 96-01-00551.

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